

When a Mathematical System Stops Changing

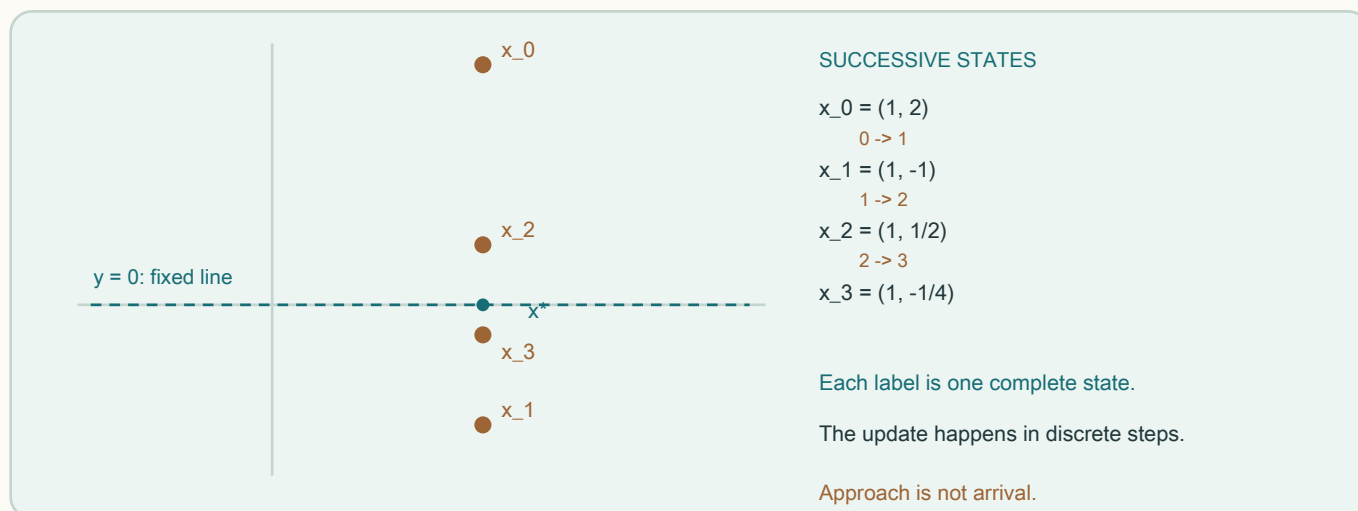
Fixed, cycling, and approaching: three different futures for a repeated rule.

THE QUESTION

If we keep feeding a rule its own output, what would it mean for the state to stop changing, and how can we tell whether it actually gets there?

Take the update rule $F(x) = Ax$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$. It leaves the horizontal coordinate unchanged and reverses and halves the vertical coordinate. Start at $x_0 = (1, 2)$. Call $x^* = (1, 0)$ a candidate stopping point; the star is just a label. We will test whether it is fixed, then follow each new output as the next input.

A fixed horizontal part and a shrinking, reversing vertical part



The horizontal coordinate stays at 1 while the vertical part alternates sides and halves. These discrete states approach the fixed state $(1, 0)$, but no finite step lands on it. Numbered arrows show step order, not positions between updates.

Work it through

Let $F(x) = Ax$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$. Start at $x_0 = (1, 2)$. Is $x^* = (1, 0)$ fixed? Where do the next three states go?

1. Test the candidate

$$F(x^*) = A(1, 0) = (1, 0) = x^*$$

The whole vector is unchanged. Since x^* is nonzero, it is an eigenvector with eigenvalue 1.

2. Take one step

$$x_1 = A(1, 2) = (1, -1)$$

The first coordinate stays 1; the second becomes -1. The state is not fixed because x_1 differs from x_0 .

3. Take two more steps

$$x_2 = A(1, -1) = (1, 1/2); x_3 = A(1, 1/2) = (1, -1/4)$$

The vertical part reverses sign and halves each time. Successive states lie on alternating sides of the fixed line $y = 0$, but none of these states is on it.

4. Separate the fixed part from the changing part

$$x_k = (1, 0) + (0, 2(-1/2)^k)$$

The first part is fixed; the second is a vertical eigenvector contribution. Its magnitude tends to zero, while its sign alternates. For any finite k , $2(-1/2)^k$ is nonzero.

The states approach $x^* = (1, 0)$, but this particular path never lands exactly on it at a finite step. 'Approaching a fixed state' and 'already fixed' describe different things.

The compact rule

A fixed state x^* satisfies $F(x^*) = x^*$. For a linear update $F(x) = Ax$, a nonzero fixed state is an eigenvector with $\lambda = 1$.

A star on x marks the state we are testing. Apply the rule F to that state. If the output is the same whole state, it is fixed. For a linear matrix rule, Ax star equals x star means one application scales this nonzero vector by one. A state can also approach a fixed state while still changing at every finite step.

Read the symbols in that rule

F and $x_{(k+1)} = F(x_k)$ - F is an update rule. Starting from x_0 , each new state is the output of applying F to the previous state. This notation describes repeated steps, not a continuous motion between them.

x^* - A candidate fixed state. The star labels a special value; it is neither multiplication nor an exponent. Check it by substituting into F .

fixed versus period two - Fixed means one step returns the same state: $F(x) = x$. A genuine two-step cycle means $F(F(x)) = x$ but $F(x) \neq x$. Returning after two steps does not by itself make a state fixed.

approach - The distance between x_k and a fixed state gets arbitrarily small as k grows. A sequence may approach its destination without equaling it at any finite step.

Compare other one-step rules

Each row is a separate possible rule, starting from v . Read across one row at a time.

Multiplier	start	once	twice	three times
1	v	v	v	v
-1	v	$-v$	v	$-v$
-1/2	v	$-v/2$	$v/4$	$-v/8$

Only multiplier 1 leaves a nonzero eigenvector unchanged after one step.

A tempting wrong turn

See one unchanged direction and declare the entire system stable, or call an alternating path fixed because it returns after two steps.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ fixes every vector on the horizontal axis but sends $(1, 1)$ to $(1, 2)$, then $(1, 4)$, so other states move away from that axis.

A factor of -1 sends a nonzero eigenvector v to $-v$ and back to v : that is period two, not fixed, because the first step changes it.

Where it appears again

A fixed point is a useful question for any repeated rule F , including nonlinear ones: solve $F(x^*) = x^*$. A matrix is a special case. In linear discrete systems, an eigenvalue of 1 marks a nonzero fixed direction, while other directions can shrink, grow, or cycle. Before making a statement about an entire system, inspect the other directions and the states under discussion. No analogy to a physical system is implied by this abstract rule alone.

One-minute reference

- Fixed: $F(x^*) = x^*$. Check by applying the rule once to the whole state.
- Period two: $F(F(x)) = x$ and $F(x) \neq x$. A nonzero eigenvector with $\lambda = -1$ does this under $F(x) = Ax$.
- For a linear rule Ax , a nonzero vector fixed by A has eigenvalue 1. The zero vector is fixed by every linear map but is not an eigenvector.
- A factor of -1/2 on an eigendirection alternates signs and shrinks toward zero; from a nonzero start, it never equals zero at finite k .
- A fixed eigendirection by itself says nothing conclusive about behavior in other directions or the stability of all starting states.

Try it yourself

Use the hints if you need a starting move. The worked answers are in the separate answer key.

006-1 / RECOGNIZE

For $F(x) = 3x - 4$ on real numbers, test whether $x^* = 2$ is fixed. Is $x = 1$ fixed?

Hint: Substitute each proposed value into F and compare output with input.

006-2 / RECOGNIZE

Let $F(x) = 2 - x$. Starting at 5 gives $5 \rightarrow -3 \rightarrow 5$. Is 5 fixed or on a genuine two-step cycle? Find one fixed state of this F .

Hint: A fixed state must survive the first arrow, not merely the second.

006-3 / APPLY

Let $x_{k+1} = A x_k$: each new state is A times the current state. For $A = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$ and $x_0 = (2, 4)$, find x_1, x_2, x_3 and the fixed state this path approaches. Does it ever reach that state in a finite number of steps?

Hint: Keep the first coordinate; multiply the second by $-1/2$ at each step.

006-4 / APPLY

A nonzero vector v satisfies $Av = -v$. Classify the repeated path $v \rightarrow Av \rightarrow A^2v$ as fixed, a genuine two-step cycle, or neither.

Hint: Apply A once to $-v$, using linearity.

006-5 / TRANSFER

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ fixes $v = (1, 0)$. Starting instead at $w = (1, 1)$, find Aw and A^2w . Does the fixed vector v prove that every state approaches a fixed state?

Hint: Compare the horizontal and vertical coordinates of w after each step.

006-6 / TRANSFER

Compare two rules with the same fixed state 7. Rule $P(x) = 7$; rule $Q(x) = 7 + (x - 7)/2$. Starting at $x_0 = 2$, which rule reaches 7 in a finite step, and which merely approaches it?

Hint: Track the error $x_k - 7$ for each rule.

Further reading

Georgia Tech Interactive Linear Algebra, §5.1 Eigenvalues and Eigenvectors: <https://textbooks.math.gatech.edu/ila/eigenvectors.html>

Georgia Tech Interactive Linear Algebra, §5.6 Stochastic Matrices: <https://textbooks.math.gatech.edu/ila/stochastic-matrices.html>

Georgia Tech Interactive Linear Algebra, §5.4 Diagonalization: <https://textbooks.math.gatech.edu/ila/diagonalization.html>