

Worked answer key / 006

When a Mathematical System Stops Changing

Each answer explains the reasoning as well as the result. The numbered prompts match the lesson and card deck.

006-1 / RECOGNIZE

For $F(x) = 3x - 4$ on real numbers, test whether $x^* = 2$ is fixed. Is $x = 1$ fixed?

2 is fixed because $F(2) = 2$. 1 is not fixed because $F(1) = -1$.

1. $F(2) = 3 \cdot 2 - 4 = 2$, so applying the rule leaves 2 unchanged.
2. $F(1) = 3 \cdot 1 - 4 = -1$, which differs from 1.
3. The fixed-point test asks for equality after one application, not just a nearby result.

006-2 / RECOGNIZE

Let $F(x) = 2 - x$. Starting at 5 gives $5 \rightarrow -3 \rightarrow 5$. Is 5 fixed or on a genuine two-step cycle? Find one fixed state of this F .

5 is on a genuine two-step cycle, not fixed. The fixed state is $x^* = 1$.

1. $F(5) = 2 - 5 = -3$, so 5 changes after one step.
2. $F(-3) = 2 - (-3) = 5$, so it returns after exactly two steps.
3. Solving $F(x) = x$ gives $2 - x = x$, hence $x = 1$; indeed $F(1) = 1$.

006-3 / APPLY

Let $x_{(k+1)} = A x_k$: each new state is A times the current state. For $A = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$ and $x_0 = (2, 4)$, find x_1, x_2, x_3 and the fixed state this path approaches. Does it ever reach that state in a finite number of steps?

$x_1 = (2, -2)$, $x_2 = (2, 1)$, $x_3 = (2, -1/2)$. They approach $(2, 0)$ but never reach it at any finite step.

1. The matrix leaves the first coordinate at 2 and multiplies the second by $-1/2$ each time.
2. The second coordinates are 4, -2, 1, $-1/2$, and in general $4(-1/2)^k$.
3. Their magnitudes tend to zero, but $4(-1/2)^k$ is nonzero for every finite k . Also $A(2, 0) = (2, 0)$, so $(2, 0)$ is fixed.

Worked answers / continued

006-4 / APPLY

A nonzero vector v satisfies $Av = -v$. Classify the repeated path $v \rightarrow Av \rightarrow A^2v$ as fixed, a genuine two-step cycle, or neither.

It is a genuine two-step cycle: $v \rightarrow -v \rightarrow v$.

1. The first step is $Av = -v$, which differs from v because v is nonzero.
2. The second is $A(-v) = -Av = -(-v) = v$.
3. Returning after two steps, with a change after one, is precisely a period-two cycle.

006-5 / TRANSFER

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ fixes $v = (1, 0)$. Starting instead at $w = (1, 1)$, find Aw and A^2w . Does the fixed vector v prove that every state approaches a fixed state?

$Aw = (1, 2)$, $A^2w = (1, 4)$. No: the vertical coordinate grows, so this starting state does not approach the fixed horizontal axis.

1. $Av = (1, 0)$ verifies v is fixed.
2. But $Aw = (1, 2)$ and $A(Aw) = (1, 4)$; the second coordinate doubles each time.
3. One fixed eigendirection describes what happens along that direction only. It cannot establish stability for all states.

006-6 / TRANSFER

Compare two rules with the same fixed state 7. Rule $P(x) = 7$; rule $Q(x) = 7 + (x - 7)/2$. Starting at $x_0 = 2$, which rule reaches 7 in a finite step, and which merely approaches it?

P reaches 7 after one step and stays there. Q approaches 7 but never reaches it from 2 in a finite number of steps.

1. Both have fixed state 7 because $P(7) = Q(7) = 7$.
2. $P(2) = 7$, so its path is $2 \rightarrow 7 \rightarrow 7 \rightarrow \dots$.
3. For Q , the initial error is $2 - 7 = -5$ and each step halves it: $x_k - 7 = -5/2^k$. This is nonzero for finite k but tends to zero.