

What Happens When You Run the Machine Again?

A matrix can act on its own output. Follow one direction, one step at a time.

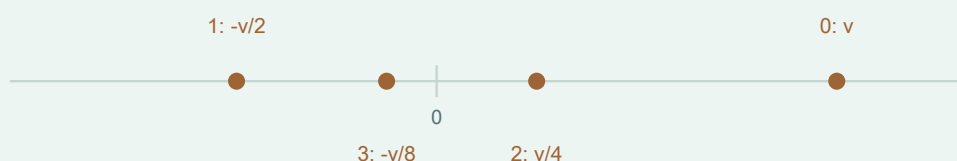
THE QUESTION

If one application of a matrix reverses an arrow and halves its length, what happens after two, three, or more applications?

Use one concrete rule first: $A = \begin{bmatrix} -1/2 & 0 \\ 0 & 2 \end{bmatrix}$, starting from $v = (1, 0)$. This matrix reverses and halves the horizontal coordinate while doubling the vertical coordinate. Because v has no vertical part, we can follow what happens when each output is fed back into A . After that, we will compare other possible one-step multipliers.

One line, an arrow that flips and shrinks

ONE PATH / REVERSE AND HALVE



Each numbered point is the next output; left of zero reverses direction.

For this chosen rule, each output moves to the other side and halves its distance from zero. Work out the numbers below.

Work it through

Let $A = \begin{bmatrix} -1/2 & 0 \\ 0 & 2 \end{bmatrix}$ and $v = (1, 0)$. What are the first three outputs, and why does the shortcut work?

1. First application

$$Av = \begin{bmatrix} -1/2 & 0 \\ 0 & 2 \end{bmatrix} (1, 0) = (-1/2, 0) = (-1/2)v$$

The result stays on the horizontal line through v . That verifies v is an eigenvector and its eigenvalue is $-1/2$.

2. Feed the output back in

$$A^2v = A(Av) = A((-1/2)v) = (-1/2)Av = (1/4)v = (1/4, 0)$$

Linearity lets the scalar come outside A . Two negative factors make a positive one.

3. One more step

$$A^3v = A((1/4)v) = (1/4)Av = (-1/8)v = (-1/8, 0)$$

The arrow reverses again and is now one eighth of its original length.

4. Read the pattern

$$v, (-1/2)v, (1/4)v, (-1/8)v, \dots; A^k v = (-1/2)^k v$$

The magnitudes $1, 1/2, 1/4, 1/8$ shrink; the signs $+, -, +, -$ alternate. No finite term is zero because $(-1/2)^k$ is never zero for finite k .

For this eigenvector, three applications give $A^3v = (-1/8, 0)$. Repeated application approaches the zero vector along this line, alternating sides.

The compact rule

At step zero, $A^0v = v$. If $Av = \lambda v$, then $A^k v = \lambda^k v$ for whole-number steps $k \geq 1$.

At step zero, no application has happened: A^0 is the identity, so the starting vector is still v . If A sends the nonzero vector v to λv , then applying A k times, for k at least one, sends v to λ raised to k times v . The exponent on A counts applications of the matrix; the exponent on λ counts how many times its scale factor is used. Separating step zero also makes the rule clear when λ is zero.

Read the symbols in that rule

A and A^k - A is a square matrix representing one linear step. A^k is A composed with itself k times: $A^2 = AA$, $A^3 = AAA$. These are matrix products, not entry-by-entry powers.

v - A nonzero eigenvector of A . The formula follows this particular direction; an arbitrary starting vector may have components that change at different rates.

λ - The scalar multiplying v in one step. Its absolute value controls the length of this eigenvector's iterates; its sign controls whether their direction reverses.

k - A whole-number step counter. It changes from 0 to 1 to 2 and onward; it is not a vector component index.

Compare other one-step rules

Each row is a separate possible rule, starting from v . Read across one row at a time.

Multiplier	start	once	twice	three times
2	v	$2v$	$4v$	$8v$
$1/2$	v	$v/2$	$v/4$	$v/8$
-2	v	$-2v$	$4v$	$-8v$
$-1/2$	v	$-v/2$	$v/4$	$-v/8$
1	v	v	v	v
-1	v	$-v$	v	$-v$
0	v	0	0	0

Each row compares a different possible one-step multiplier, starting from a vector called v . The absolute value of each coefficient gives the arrow's length relative to v ; a minus sign means its direction has reversed.

A tempting wrong turn

To find A^2 , square every number inside A , or assume $A^k w = \lambda^k w$ for any vector w .

A^2 means AA . For example, $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ gives $M^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, while squaring each entry would leave M unchanged. The λ^k shortcut applies to a verified eigenvector v . A vector made from several directions can carry several scale factors at once.

Where it appears again

Discrete linear models repeat a rule of the form $x_{k+1} = Ax_k$. If the initial state is an eigenvector, this lesson gives its exact state after k steps. For a general state, more work is needed to account for all its directions; a single eigenvalue is not a verdict about the whole system.

One-minute reference

- Check $Av = \lambda v$ with $v \neq 0$ before using the eigenvector shortcut.
- A^k means k applications of A ; A^0 is the identity. Step zero is v , even for $\lambda = 0$, because no multiplication has happened yet.
- For whole-number steps $k \geq 1$ on an eigenvector, $A^k v = \lambda^k v$. This follows by repeatedly applying $Av = \lambda v$; step zero is v .
- For real λ , $|\lambda|$ controls size: above 1 grows, between 0 and 1 shrinks, equal to 1 keeps the same size. A negative λ also reverses direction each step.
- $\lambda = 0$ sends an eigenvector to zero in one step; a nonzero λ with $|\lambda| < 1$ approaches zero without reaching it in a finite number of steps.

Try it yourself

Use the hints if you need a starting move. The worked answers are in the separate answer key.

005-1 / RECOGNIZE

Suppose $Av = 3v$ for a nonzero vector v . Read A^4v aloud, then state its value as a multiple of v .

Hint: Count four applications of A , starting from v .

005-2 / RECOGNIZE

A nonzero eigenvector v has $\lambda = -2$. List its positions at steps $k = 0, 1, 2, 3$ as multiples of v . Does its length grow or shrink with each application, and does it point to the same side of the origin?

Hint: Track absolute value and sign separately: $(-2)^0, (-2)^1, (-2)^2, (-2)^3$.

005-3 / APPLY

A nonzero eigenvector v has $\lambda = -1/2$. Write its positions at steps $k = 0, 1, 2, 3$, and 4 as multiples of v . Does any of these steps land on zero?

Hint: Multiply the previous coefficient by $-1/2$ each time.

005-4 / APPLY

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $v = (0, 2)$. Verify $Av = \lambda v$, identify λ , and find A^2v .

Hint: Multiply A by v once; compare the entire output with v .

005-5 / TRANSFER

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $w = (1, 1)$. Compute Aw and A^2w . Why can you not choose one λ and write $A^2w = \lambda^2w$ here?

Hint: Look at what happens to the first and second coordinates separately after one step.

005-6 / TRANSFER

For $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, compare the true matrix square M^2 with the array obtained by squaring each entry. Are they equal?

Hint: The top-right entry of MM is $\text{row } 1 \cdot \text{column } 2$.

Further reading

Georgia Tech Interactive Linear Algebra, §5.1 Eigenvalues and Eigenvectors: <https://textbooks.math.gatech.edu/ila/eigenvectors.html>

Georgia Tech Interactive Linear Algebra, §5.4 Diagonalization: <https://textbooks.math.gatech.edu/ila/diagonalization.html>

Georgia Tech Interactive Linear Algebra, §5.6 Stochastic Matrices: <https://textbooks.math.gatech.edu/ila/stochastic-matrices.html>