

The Directions a Transformation Doesn't Turn

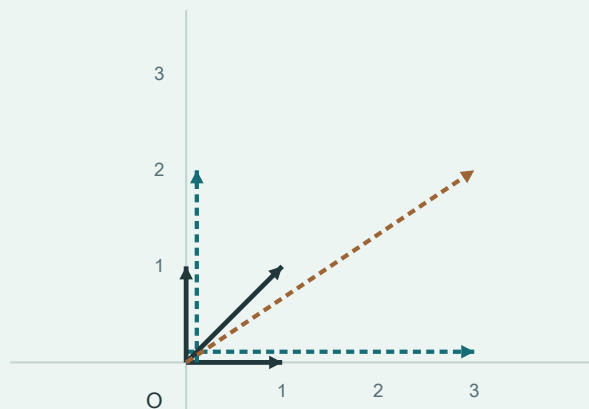
Eigenvectors and eigenvalues, read as an instruction

THE QUESTION

When a matrix moves an arrow, is there a line that the arrow stays on?

Start with $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, a two-by-two matrix that triples an arrow's horizontal coordinate and doubles its vertical coordinate. Try three input arrows: $v = (1, 0)$ points right, $w = (0, 1)$ points up, and $u = (1, 1)$ points diagonally. We will check whether each output lies on the same line through the origin as its input.

Three arrows through the same grid



solid: input

dashed: output after A

$(1, 0) \rightarrow (3, 0)$

$(0, 1) \rightarrow (0, 2)$

$(1, 1) \rightarrow (3, 2)$

The diagonal leaves its line.

Under $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, the horizontal and vertical input arrows stay on their own lines. The diagonal input moves off its original line.

Work it through

For $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, compare $v = (1, 0)$, $w = (0, 1)$, and $u = (1, 1)$. Which inputs are eigenvectors?

1. Start with the first arrow

$$A(1, 0) = (3 \times 1 + 0 \times 0, 0 \times 1 + 2 \times 0)$$

The first row makes the first output coordinate; the second row makes the second.

2. Compare it with a scaled copy

$$A(1, 0) = (3, 0) = 3(1, 0)$$

One number, 3, multiplies both coordinates. v stays on the horizontal line, so v is an eigenvector with eigenvalue 3. Its length changes; its line does not.

3. Check the vertical arrow

$$A(0, 1) = (3 \times 0 + 0 \times 1, 0 \times 0 + 2 \times 1) = (0, 2) = 2(0, 1)$$

w stays on the vertical line. It is an eigenvector with eigenvalue 2.

4. Test the diagonal arrow

$$A(1, 1) = (3, 2)$$

To equal $\lambda(1, 1) = (\lambda, \lambda)$, both output coordinates would need to be the same number. They are 3 and 2. No one scalar works, so u is not an eigenvector.

The horizontal and vertical inputs keep their respective lines, with factors 3 and 2. The diagonal input moves off its line.

The compact rule

$$Av = \lambda v, \text{ with } v \neq 0$$

Apply the square matrix A to the nonzero vector v . The answer is the same vector multiplied by one number, λ (lambda). If that happens, v is an eigenvector of A and λ is its eigenvalue.

Read the symbols in that rule

A - A square matrix: a rule that takes a vector to another vector with the same number of coordinates. In this note, A has two rows and two columns.

v - A nonzero input vector. We write its two coordinates as (x, y) , shorthand for a two-entry column vector. The zero vector cannot be an eigenvector.

Av - The output after the matrix acts on v . Matrix-vector multiplication gives each output coordinate from a row of A .

λv - One scalar λ multiplies every coordinate of v . λ is a number, not a second matrix.

A tempting wrong turn

A is diagonal, so every input must keep its direction.

The two coordinate directions are scaled by different factors. For $(1, 1)$, A gives $(3, 2)$, which is not a scaled copy of $(1, 1)$. An eigenvector needs one common factor for all its coordinates.

Where it appears again

In the next note, applying A repeatedly to an eigenvector gives $A^n v = \lambda^n v$. The vector keeps its line while the scalar factor changes with each application. That statement concerns this one eigenvector; it does not by itself describe every possible input.

One-minute reference

- Test a candidate: multiply Av , then ask whether one scalar λ makes every coordinate of Av equal to λ times the corresponding coordinate of v .
- v must be nonzero and A must be square. λ may be zero: then Av is the zero vector, which has no direction.
- Positive λ keeps the arrow's orientation; negative λ reverses it. $\lambda = 1$ leaves the vector unchanged. The factor $|\lambda|$ controls its length when λ is nonzero.
- An eigenvector stays on its line through the origin; it need not keep the same length. Not every real matrix has a real eigenvector: a 90° plane rotation is an example.

Try it yourself

Use the hints if you need a starting move. The worked answers are in the separate answer key.

004-1 / RECOGNIZE

A square matrix gives $Av = v$ for a nonzero v . Is v an eigenvector? What is λ ?

Hint: Compare Av with λv . Which number leaves every coordinate as it was?

004-2 / RECOGNIZE

For $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, someone says $(1, 1)$ is an eigenvector because $A(1, 1) = (3, 2)$ still points generally up and right. Check the claim.

Hint: Could $(3, 2)$ equal $\lambda(1, 1)$ for one number λ ?

004-3 / APPLY

Keep $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Test $v = (0, -3)$. Give Av and the eigenvalue if v qualifies.

Hint: Use the two rows of A . Then compare the output with a multiple of $(0, -3)$.

004-4 / APPLY

Let $D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ and $v = (3, 0)$. Find Dv , decide whether v is an eigenvector, and describe what the sign of λ does.

Hint: Find Dv first. A negative scalar puts the output on which side of the origin?

004-5 / TRANSFER

Let $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $v = (2, 0)$. P sends v to the origin. Does v count as an eigenvector? Would the zero vector itself count?

Hint: The definition restricts the input v , not the output Pv or the value of λ .

004-6 / TRANSFER

$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every plane arrow 90° counterclockwise. Test $(1, 0)$, then decide whether any nonzero real vector can be an eigenvector of R .

Hint: A real scalar multiple stays on the input's line, reverses along it, or is zero. What does a quarter-turn do?

Further reading

Georgia Tech, Interactive Linear Algebra, §5.1 Eigenvalues and Eigenvectors: <https://textbooks.math.gatech.edu/ila/eigenvectors.html>

Georgia Tech, Interactive Linear Algebra, §5.5 Complex Eigenvalues: <https://textbooks.math.gatech.edu/ila/complex-eigenvalues.html>