

Rows, Columns, and Why You Don't Multiply Everything

Matrix multiplication: from addresses to answers

THE QUESTION

If we multiply A by B, how do we find the numbers in $C = AB$?

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$. Multiply A by B to find every entry of $C = AB$. We will choose one entry, work out how it is made, and then fill the rest.

Name the objects

A and B - Whole matrices. A has 2 rows and 3 columns; B has 3 rows and 2 columns. Their shared dimension is 3.

$C = AB$ - The whole output matrix. Its outside dimensions are 2 rows and 2 columns, so C has four entries to calculate.

c_{ij} - One output entry. The first subscript i selects its row; the second subscript j selects its column.

Check the shapes

$$C = AB; (2 \times 3)(3 \times 2) \rightarrow 2 \times 2$$

One address chooses one row and one column

A / 2 rows x 3 columns

2	1	3
4	2	1

A row 2: (4, 2, 1)

B / 3 rows x 2 columns

1	2
3	1
2	4

B column 1: (1, 3, 2)

$C = AB$ / 2 x 2

?	?
?	?

c_{21} : still unknown

Match positions: 4 with 1, 2 with 3, 1 with 2.

We begin at C's lower-left cell, row 2 and column 1. Its address is c_{21} . Mark the horizontal second row of A, the upright first column of B, and that still-unknown cell of C. Labels and strip shapes carry the distinction alongside color.

The number at row two, column one of C comes from row two of A and column one of B. Pair their first numbers, then their second, then their third; multiply within each pair and add the three products.

Work it through

Multiply $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$ by $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$. Find all four entries of $C = AB$. We will begin with its lower-left cell, then fill the other three.

1. Check the shapes

$$(2 \times 3)(3 \times 2) \rightarrow 2 \times 2$$

The inside 3 matches. The outside numbers describe the output shape, so four entries need to be made. The shared 3 gives the number of products within each entry.

2. Choose one address

c_{21} : A row 2 = (4, 2, 1); B column 1 = (1, 3, 2)

The lower-left cell of C is in row 2, column 1, so we call it c_{21} . Any of the four cells could come first. For this one, match the first positions, second positions, and third positions.

3. Expand and calculate

$$c_{21} = 4 \times 1 + 2 \times 3 + 1 \times 2 = 4 + 6 + 2 = 12$$

Three corresponding pairs give one number. This is one entry of C, not yet the whole matrix.

Each output address, expanded

$$c_{11} = 2 \times 1 + 1 \times 3 + 3 \times 2 = 2 + 3 + 6 = 11$$

$$c_{12} = 2 \times 2 + 1 \times 1 + 3 \times 4 = 4 + 1 + 12 = 17$$

$$c_{21} = 4 \times 1 + 2 \times 3 + 1 \times 2 = 4 + 6 + 2 = 12$$

$$c_{22} = 4 \times 2 + 2 \times 1 + 1 \times 4 = 8 + 2 + 4 = 14$$

4. Fill the other addresses

$$c_{11}=11; c_{12}=17; c_{22}=14; C=[[11, 17], [12, 14]]$$

Repeat the row-column operation for each remaining output address, then place the four results in their positions.

$AB = [[11, 17], [12, 14]]$. Each entry is a sum of three matched products; the matrix is the assembly of those four entries.

The compact rule after the arithmetic

For any compatible A and B, fix the output row i and column j. Let z run through the n matching positions, multiply the entry of A at row i, column z by the entry of B at row z, column j, and add those products. This gives one entry c_{ij} . Repeat for every output address to assemble C.

$$c_{ij} = \sum_{z=1}^n a_{iz} b_{zj}$$

A tempting wrong turn

Multiply every number in the chosen row by every number in the chosen column, making nine products for one entry.

The multiplication rule pairs corresponding positions. For c_{21} , position 1 gives 4×1 , position 2 gives 2×3 , and position 3 gives 1×2 . The shared dimension 3 counts those products per entry. It does not count output entries, and it does not ask for all nine row-column combinations.

Where it appears again

A matrix product can describe the composition of transformations. In a later note, a square matrix A and nonzero eigenvector v satisfy $Av = \lambda v$ for some scalar λ . Step zero leaves v unchanged; for whole-number steps $n \geq 1$, repeated application gives $A^n v = \lambda^n v$. This is repeated matrix application, not entrywise squaring. A real matrix need not have a real eigenvector.

One-minute reference

- Check shape: the adjacent dimensions must match. $(2 \times 3)(3 \times 2)$ gives a 2×2 output.
- Choose one address c_{ij} : i fixes the row of A, and j fixes the column of B.
- Pair position 1 with 1, position 2 with 2, and so on through the shared dimension. Multiply within each pair and add.
- One sum gives one c_{ij} . Repeat over all valid output addresses to assemble C.

Try it yourself

Use the hints if you need a starting move. The worked answers are in the separate answer key.

002-1 / RECOGNIZE

In $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$, find a_{23} and b_{32} . Say each address as row, column before reading its value.

Hint: For a_{23} , use A's second row and third column. For b_{32} , use B's third row and second column.

002-2 / RECOGNIZE

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$, and $C = AB$. For c_{12} , which row of A and column of B must meet? How many paired products make that one entry?

Hint: Read 12 as row 1, column 2. The shared dimension is 3.

002-3 / APPLY

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$, and $C = AB$. Calculate c_{22} . Write the chosen row and column before the arithmetic.

Hint: Take A row 2, (4, 2, 1), and B column 2, (2, 1, 4).

002-4 / APPLY

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$.

Is BA defined? If so, give its shape and top-left entry.

Hint: B is 3×2 and A is 2×3 . For the top-left entry, use B row 1 and A column 1.

002-5 / APPLY

Let $P = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Find PQ and its shape.

Hint: Q has one column, so PQ has one column. Make its two entries using each row of P with that column.

002-6 / TRANSFER

A is 2×3 , B is 3×2 , and D is any 2×3 matrix. Which of AB , BA , AD , and DA are defined? Give each valid output shape.

Hint: For each order, compare only the adjacent dimensions. Then read the outside dimensions if they match.

002-7 / TRANSFER

Without calculating entries, describe $(4 \times 9)(9 \times 6)$: output shape, total output entries, and paired products per entry.

Hint: The outside numbers make the output shape. Count its cells, then use the shared dimension for the number of products inside each cell.

002-8 / TRANSFER

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$, and $C = AB$. Someone tries nine every-to-every products for c_{21} . Which row and column should they choose, how should they pair positions instead, and what is c_{21} ?

Hint: Write the two three-number strips side by side. Pair only entries at the same position.

Further reading

Georgia Tech, Interactive Linear Algebra, §3.4 Matrix Multiplication: <https://textbooks.math.gatech.edu/ila/1553/matrix-multiplication.html>

Georgia Tech, Interactive Linear Algebra, §5.1 Eigenvalues and Eigenvectors: <https://textbooks.math.gatech.edu/ila/1553/eigenvectors.html>

Georgia Tech, Interactive Linear Algebra, §5.5 Complex Eigenvalues: <https://textbooks.math.gatech.edu/ila/1553/complex-eigenvalues.html>